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Continuum Moment Equations on the Lattice

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Abstract

An analysis is given as to why one can not directly evaluate continuum moment equations, i.e., equations involving powers of the position variable times charge, current, or energy/momentum operators, on the lattice. I examine two cases: a three point function evaluation of the nucleon magnetic moment and a four point function (charge overlap) evaluation of the pseudoscalar charge radius.

I. INTRODUCTION

Beautiful hadronic moment equations can be derived from continuum field theory three and four point matrix elements[1]. It is very tempting to try to implement these equations directly in lattice simulations. A number of papers in fact have used them, assumed them to be true, or examined their consequences in lattice calculations[2, 3, 4, 5]. Physical quantities considered have been charge radii[2, 5], magnetic moments[3, 4] and quark total angular momentum [3]. However, these type of equations share one crucial feature in their derivation: a derivative with respect to momentum transfer evaluated at zero momentum. This last step can not be reproduced on the lattice because of the finite momentums available there, so the question arises as to the validity of such continuum-derived expressions evalauted on the lattice. We will examine two such expressions in this paper and will see that the continuum expectations and the lattice reality can differ markedly.

II. TWO EXAMPLES

Let us recap the situation for one such specific case, an expression for nucleon magnetic moments given in Refs.[3, 4]. Both two and three point functions appear in this expression. The time ordered two point function, using the proton interpolation field, $\chi_\alpha^p(x)$, is (understood α, α' sums)

$$G_{pp}(t; \vec{p}, \Gamma) \equiv \sum_{\vec{x}} e^{-i\vec{p} \cdot \vec{x}} \Gamma_{\alpha' \alpha} \langle \text{vac} | T (\chi_\alpha^p(x) \overline{\chi}_{\alpha'}^p(0)) | \text{vac} \rangle, \quad (1)$$

where $\Gamma_4 \equiv \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\Gamma_k \equiv \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}$. The long Euclidean time limit of this expression for $\Gamma = \Gamma_4$ is given by

$$G_{pp}(t; \vec{p}, \Gamma_4) \xrightarrow{t \gg 1} \frac{E_p + m_N}{2E_p} \frac{|Z|^2 a^6}{(2\kappa)^3} e^{-E_p t}, \quad (2)$$

where m_N is the (dimensionless) nucleon mass, “a” is the lattice spacing, and $\kappa = 1/(2(m + 4))$, In addition, Z is the normalization factor, $(\text{vac} | \chi_\alpha^{p;cont}(0) | \vec{p}, s) = Z u_\alpha(\vec{p}, s)$, where $u_\alpha(\vec{p}, s)$ is the free Dirac spinor, $|\vec{p}, s)$ is a continuum proton state, and $\chi_\alpha^{p;cont}(x)$ is the continuum interpolation field. The three point function we need, which uses the conserved vector current, $J_\mu(x)$, is

$$G_{pJ_\mu p}(t_2, t_1; \vec{p}, \vec{p}', \Gamma) \equiv -i \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p} \cdot \vec{x}_2} e^{i\vec{q} \cdot \vec{x}_1} \Gamma_{\alpha' \alpha} \langle \text{vac} | T (\chi_\alpha^p(x_2) J_\mu(x_1) \overline{\chi}_{\alpha'}^p(0)) | \text{vac} \rangle, \quad (3)$$

which has the long Euclidean time limit,

$$G_{pJ_{jp}}(t_2, t_1; 0, -\vec{q}, \Gamma_k) \xrightarrow{(t_2-t_1), t_1 \gg 1} \frac{1}{2E_\ell} \frac{|Z|^2 a^6}{(2\kappa)^3} e^{-m_N(t_2-t_1)} e^{-E_\ell t_1} \epsilon_{ijk}(\vec{q}_\ell)_i G_m(Q_\ell^2). \quad (4)$$

The Minkowski four momentum transfer squared is given by $Q_\ell^2 = 2m_N(E_\ell - m_N)$, we are assuming continuum dispersion, $E_\ell^2 = m_N^2 + \vec{q}_\ell^2$, and $(\vec{q}_\ell)_i = \frac{\pi \ell}{N}$, $\ell = 0, \pm 1, \pm 2, \dots, N$ in a given momentum direction for a square spatial lattice of size $N_s = (2N)^3$. (All these considerations can be reformulated for a lattice with an odd number of spatial sites in given directions, but no fundamentally different conclusions or observations results.) Taking a “continuum” derivative of $G_{pJ_{jp}}(t_2, t_1; 0, -\vec{q}, \Gamma_k)$ with respect to $(\vec{q}_\ell)_i$, evaluated at zero momentum, gives us another three point function, which we will define as

$$G_{pJ_{jp}}(t_2, t_1; (\vec{x}_1)_i, \Gamma_k) \equiv \sum_{\vec{x}_2, \vec{x}_1} (\Gamma_k)_{\alpha'\alpha} \langle \text{vac} | \chi_\alpha^p(x_2) (\vec{x}_1)_i J_j(x_1) \bar{\chi}_{\alpha'}^p(0) | \text{vac} \rangle. \quad (5)$$

Setting this quantity equal to the continuum derivative of Eq.(4) then results in

$$\left. \frac{G_{pJ_{jp}}(t_2, t_1; (\vec{x}_1)_i, \Gamma_k)}{G_{pp}(t_2; \vec{p}, \Gamma_4)} \right|_D \xrightarrow{(t_2-t_1), t_1 \gg 1} \epsilon_{ijk} \frac{G_m(0)}{2m_N}. \quad (6)$$

The “D” notation reminds us that this result follows from using a momentum derivative evaluated at zero momentum.

Ref.[4] justified this procedure in the following way. One can imagine first taking the spatial momentum derivative of the continuum analog of $G_{pJ_{jp}}(t_2, t_1; 0, -\vec{q}, \Gamma_k)$, evaluated at $\vec{q} = 0$, and dividing by the continuum two point function. One then transcribes this result into lattice language by changing the continuum matrix elements into lattice ones and making the appropriate substitutions for the spatial integration and the various fields. It was found there that this procedure resulted in a lattice measurement giving unrealistically small neutron and proton magnetic moments, $G_m(0)$. In particular, there was a downward trend in the data for smaller quark mass. The results in [3] are similar, although the values are larger.

In order to understand why this equation fails on the lattice, let us expand the position variable in terms of the momentum eigenstates of the periodic lattice. In one dimension we have the discrete completeness statement

$$\frac{1}{2N} \sum_{n=-N+1}^N e^{-iq\ell n} = \delta_{\ell,0}. \quad (7)$$

On a periodic lattice with an even number of sites, choosing an origin forces one side or the other of the lattice to have one extra spatial site. For this purpose, one needs to define a function which represents a linear function everywhere except at the extra site, $n = N$, where, because it may be considered equally distant from the origin in either direction, we will take it to be zero. (This is the same function used in the numerical evaluations in [3, 4].) Thus,

$$F(n) = \begin{cases} n & , n \neq N \\ 0 & , n = N \end{cases} \quad (8)$$

We expand this in terms of the momentum eigenfunctions, $e^{iq_\ell n}$,

$$F(n) = \sum_{l=-N+1}^N C_\ell e^{iq_\ell n}. \quad (9)$$

Using Eq.(7), this gives

$$C_\ell = \frac{1}{2N} \sum_{n=-N+1}^{N-1} F(n) e^{-iq_\ell n}. \quad (10)$$

Summing this finite series in the usual way by multiplying both sides by a phase factor, e^{-iq_ℓ} , and shifting the summation limits (note that $e^{\pm iq_\ell N} = (-1)^\ell$), one finds that

$$C_\ell = \begin{cases} \frac{i}{2}(-1)^\ell \cot(q_\ell/2) & , \ell \neq 0 \\ 0 & , \ell = 0 \end{cases} \quad (11)$$

This leads to

$$\left. \frac{G_{pJ_p}(t_2, t_1; (\vec{x}_1)_i, \Gamma_k)}{G_{pp}(t_2; \vec{p}, \Gamma_4)} \right|_S \xrightarrow{(t_2-t_1), t_1 \gg 1} -\frac{e^{m_N t_1}}{4m_N} \epsilon_{ijk} \sum_{\ell \neq 0} (-1)^\ell \cot(q_\ell/2) \frac{e^{-E_\ell t_1}}{E_\ell} (\vec{q}_\ell)_i G_m(Q_\ell^2). \quad (12)$$

The “S” notation now reminds us that this result follows from explicitly performing the lattice sum. The coordinate $(\vec{x}_1)_i$, when inserted in Eq.(5), projects over the lattice momentums with a function given by Eq.(11).

The leading terms in Eq.(12) define what we will term the extreme Euclidean time limit (EETL) on the t_1 variable as,

$$\left. \frac{G_{pJ_p}(t_2, t_1; (\vec{x}_1)_i, \Gamma_k)}{G_{pp}(t_2; \vec{p}, \Gamma_4)} \right|_S \xrightarrow{EETL} \epsilon_{ijk} \frac{e^{(m_N - E_1) t_1}}{E_1} G_m(Q_1^2), \quad (13)$$

where we have approximated $\sin((\vec{q}_1)_i/2) \approx (\vec{q}_1)_i/2$. Eq.(13) is not suitable to measure the magnetic moment on the lattice. At fixed finite $(\vec{q}_1)_i$, the signal involves only the lowest nonzero component of $G_m(Q_1^2)$, and is not time independent.

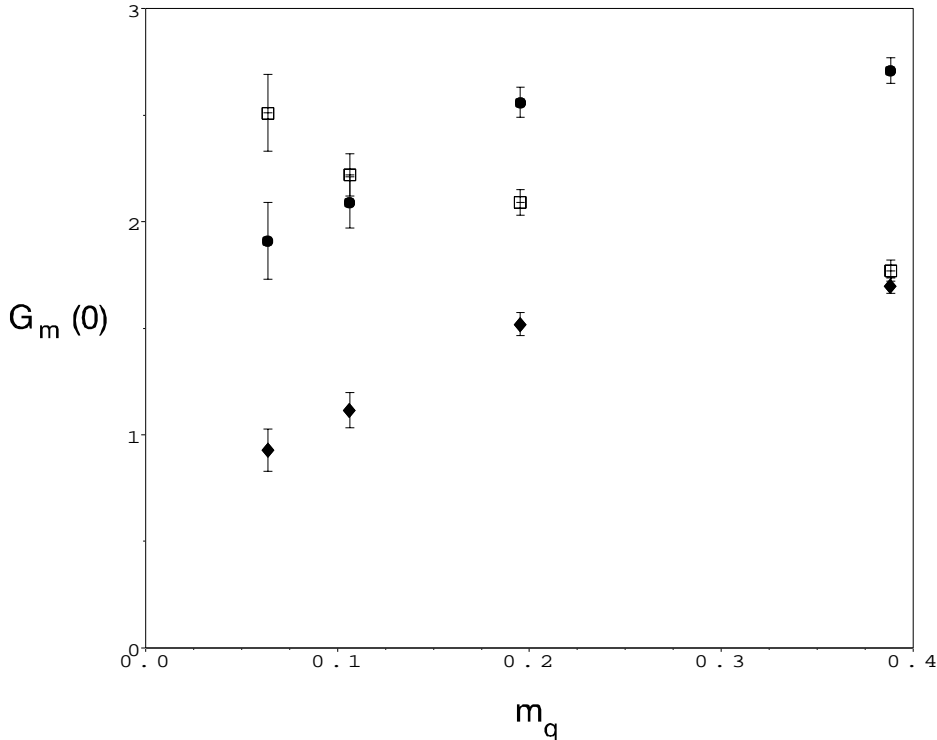


FIG. 1: The magnetic moment of the proton at four values of m_q , extracted three different ways on a $16^3 \times 24$ lattice from the data of Ref.[4]. Square symbols are extrapolated zero momentum form factors. The diamonds are extracted values assuming Eq.(6) were valid. Circles show the result one would find for Eq.(6) if the lowest momentum EETL limit in Eq.(13) was dominant in the momentum sum in Eq.(12).

It is not a contradiction that Eqs.(6) and (13) disagree with one another even in the $q_1 \rightarrow 0$ (or $N \rightarrow \infty$) limit. If one formed a discrete *lattice* derivative for the left side of Eq.(6), equivalent to simply evaluating Eq.(4) at the lowest spatial momentum and dividing by that momentum, one would obtain a result consistent with (6) in the $q_1 \rightarrow 0$ limit. Of course, one may always use external field methods to consistently extract magnetic moments. Then one is effectively taking derivatives with respect to the external field rather than momentum to isolate the coupling.

Fig. 1 shows measurements of the lattice proton magnetic form factor at zero momentum transfer on a $16^3 \times 24$ lattice. The open square symbols are extrapolated from nonzero momentum[4]. A measurement assuming Eq.(6) yields the solid diamond symbols, which are trending downward as a function of decreasing quark mass. The results in Ref.[3] are

similar. Using Eq.(13) one can test the extent to which the lowest momentum is contributing to the result from (12) because one has separate data for $G_m(Q_1^2)$ at the lowest momentum transfer[8]. The solid circles show the result one would find for Eq.(6) if the lowest momentum EETL limit in Eq.(13) was dominant in the momentum sum. The trend is also downward for smaller quark mass, which is a result of the decreasing value of $G_m(Q_1^2)$ as well as the greater exponential suppression from the $e^{(m_N-E)t}$ factor. ($t = 7\frac{1}{2}$ for the data from Ref.[4]). The qualitative behaviors are remarkably similar for these two Eq.(6) measurements, although significant cancellation is probably occurring for the diamond data from the $(-1)^\ell$ factor in Eq.(12). It was speculated in [3, 4] that the downward trend possibly reflected the fact that the nucleon was not well contained in the lattice volume. The Fig. 1 data strongly suggests that the downward trend in the diamond symbols is instead the result of two factors, decreasing $G_m(Q_\ell^2)$ and greater $e^{(m_N-E)t}$ suppression, similar to the EETL case[9].

As another example of a continuum moment equation on the lattice, consider the charge overlap measurement of the pion charge radius. Time separated measurements of charge overlap matrix elements were first considered in lattice calculations in Ref.[6].

We will start with the results derived in Ref[7]. Using u, d flavor conserved lattice charge densities, $\rho^{u,d}(x)$, one has for $t_3 \gg t_{1,2} \gg 1$,

$$\frac{\sum_{\vec{x}_3, \vec{z}} \langle 0 | \phi^\dagger(x_3) T(\rho^u(x_2) \rho^d(x_1)) \phi(z) | 0 \rangle}{\sum_{\vec{x}_3, \vec{z}} \langle 0 | \phi^\dagger(x_3) \phi(z) | 0 \rangle} \longrightarrow \mathcal{P}_\pi^{ud}(\vec{r}, t), \quad (14)$$

where $z = (\vec{z}, 0)$ and ϕ is a charged pion interpolation field. $\mathcal{P}_\pi^{ud}(\vec{r}, t)$ can be written using space and time translational invariance between the zero momentum pion states ($\vec{r} \equiv \vec{x}_2 - \vec{x}_1, t \equiv t_2 - t_1$) as

$$\mathcal{P}_\pi^{ud}(\vec{r}, t) = \langle \pi(\vec{0}) | T(\rho^u(r) \rho^d(0)) | \pi(\vec{0}) \rangle. \quad (15)$$

The discrete Fourier transform is defined to be,

$$\mathcal{Q}(\vec{q}^2, t) \equiv N_s \sum_{\vec{r}} e^{-i\vec{q} \cdot \vec{r}} \mathcal{P}_\pi^{ud}(\vec{r}, t). \quad (16)$$

It is important to let $t \gg 1$ in Euclidean space in order to damp out the contributions of higher mass intermediate states when a complete set of states is inserted between the charge densities in (15). One then has

$$\mathcal{Q}(\vec{q}^2, t) \xrightarrow{t \gg 1} \frac{(E_q + m_\pi)^2}{4E_q m_\pi} F_\pi^2(Q^2) e^{(m_\pi - E_q)t}, \quad (17)$$

where $Q^2 = 2m_\pi(E_q - m_\pi)$ and $F_\pi(Q^2)$ is the pion form factor.

A continuum derivative of Eq.(16) with respect to \vec{q}^2 at zero momentum forms the quantity

$$\mathcal{R}_\pi^2(t) \equiv N_s \sum_{\vec{r}} \vec{r}^2 \mathcal{P}_\pi^{ud}(\vec{r}, t). \quad (18)$$

Using Eqs.(16),(17) and (18) and following a procedure similar to the above for the magnetic moment, we obtain the continuum derivative result,

$$\mathcal{R}_\pi^2(t) \Big|_D \xrightarrow{t \gg 1} 2R_{u,d}^2 + \frac{3t}{m_\pi}, \quad (19)$$

where $R_{u,d}^2$ is the charged pion u, d quark charge radius. This is the same result as in Ref.[5] when the bag sources there are replaced with zero momentum pion sources. One could imagine evaluating this expression on the lattice to try to extract the charge radius from the time constant term, but we will see this hope is ill-founded.

In order to explain why one can not measure the charge radius from an expression like Eq.(18), let us now expand the square of the position variable, \vec{r}^2 , in terms of the momentum eigenstates. For a one dimensional lattice we will consider,

$$n^2 = \sum_{\ell=-N+1}^N K_\ell e^{iq_\ell n}. \quad (20)$$

We then find that the coefficients, K_ℓ , are given by

$$K_\ell = \begin{cases} \frac{1}{2}(-1)^\ell \csc^2(q_\ell/2) & , \ell \neq 0 \\ \frac{1}{3}(N^2 + \frac{1}{2}) & , \ell = 0 \end{cases} \quad (21)$$

In a three dimensional context, the quantities on the right in (21) are multiplied by zero momentum Kronecker deltas in the transverse directions.

Using Eqs.(20) and (21) in (18) and inserting a complete set, one finds that

$$\mathcal{R}_\pi^2(t) \Big|_S \xrightarrow{t \gg 1} (N^2 + \frac{1}{2}) + \frac{3}{2} \sum_{\ell \neq 0} (-1)^\ell \csc^2(q_\ell/2) \frac{(E_\ell + m_\pi)^2}{4m_\pi E_\ell} F_\pi^2(Q_\ell^2) e^{(m_\pi - E_\ell)t}. \quad (22)$$

We can define a discrete *lattice* charge radius as

$$(R_{u,d}^2)_L \equiv 3 \frac{(1 - F_\pi^2(Q_1^2))}{q_1^2}. \quad (23)$$

In the EETL, we then find,

$$\mathcal{R}_\pi^2(t) \Big|_S \xrightarrow{EETL} (N^2 + \frac{1}{2}) - 12 \left(\frac{N^2}{\pi^2} - \frac{1}{3} (R_{u,d}^2)_L \right) e^{(m_\pi - E_1)t}. \quad (24)$$

So we see that in contrast to Eq.(19), the first (time independent) term is essentially meaningless. It is conceivable that one could extract a measurement of $(R_{u,d}^2)_L$ from the second, time dependent term, but it would be much simpler to project out the form factor at the lowest lattice momentum in the usual manner and use Eq.(23). Again, the fact that (19) and (24) do not agree even for very fine lattices ($N \rightarrow \infty$) is not a contradiction because one has not taken a discrete lattice derivative, but a continuum one in producing (19).

III. SUMMARY

We have seen why it is not possible to directly evaluate continuum moment equations on a periodic lattice. Continuum moments of lattice operators in a periodic system do not project onto good momentum and so do not isolate low momentum properties. The present author pointed this fact out some years ago in the context of charged pion polarizability calculations[7]. In a more general sense, the lesson we have learned is that in order to deduce continuum properties from the lattice, it is important to treat the lattice as a self-consistent physical system. Position functions are meaningful only if expanded in terms of the available momentum eigenstates of the system. It can be very misleading in general to try to take continuum field theory equations and simply “lattice” them in deducing physical properties.

IV. ACKNOWLEDGMENTS

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